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LETTER TO THE EDITOR

Critical KAM circles and the Brjuno function

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Abstract. We consider a complex Hamiltonian map, and show how to approximate the critical value of the perturbation parameter at which a given KAM circle disappears by means of a function (the Brjuno function) which only depends on the continued fraction expansion of the rotation number.

The existence of invariant circles in the dynamics of iterated area-preserving maps has been intensively investigated by various authors (Arnol'd 1961, Aubry 1983, Celletti and Chierchia 1988, Greene 1979, Greene and Percival 1981, Herman 1983, 1986, Mackay 1983, Mackay and Percival 1985, Mather 1984, Moser 1962, 1986).

From Kolmogorov-Arnol'd-Moser (KAM) theory, one knows that most invariant circles are preserved under small perturbations of integrable maps. On these invariant circles the dynamics is analytically conjugated to translations with rotation numbers which are strongly irrational, so as to verify, for example, some Diophantine inequality. For large enough perturbations it has been numerically and analytically shown that KAM circles disappear.

The problem of obtaining accurate estimates of the breakdown threshold for an invariant circle of given rotation number is still basically unsolved, since one lacks a rigorous and computationally effective method.

On the other hand, it is generally believed that the value of the parameter at which the perturbation series for a given invariant circle diverges coincides with the breakdown threshold, and at least it certainly gives an extremely good lower bound.

The former is, for instance, the case for complex analytic maps, where the existence of critical points on the breakdown circles allows one to study the dependence of this threshold on the rotation number with a very good accuracy (Marmi 1988a, b, 1989).

We report here briefly on a simple number-theoretical function, the Brjuno function (Brjuno 1971, 1972, Yoccoz 1988), which provides a natural tool for estimating this threshold for complex analytic and area-preserving maps. We refer to Marmi (1989) for more details and proofs.

Given an area-preserving map, let K denote the perturbation parameter, so that when $K = 0$ the phase space is completely foliated into invariant circles. The critical value of K, at which an invariant circle is destroyed, clearly depends on the rotation number ω and on the map itself.

Critical functions $K = K(\omega)$ were first studied by Percival (1982) for the semistandard map, which is a so-called half-plane map. These are complex area-preserving maps, but show most of the relevant features of real maps. More recently, Percival

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and Vivaldi (1988) have studied the critical function for the modulated singular map (MSM)

$$p_{n+1}=p_n+\frac{K\exp(2\pi i n\omega)}{q_n-1} \qquad q_{n+1}=q_n+p_{n+1} \tag{1}$$

where one considers invariant circles with rotation number equal to the frequency ω of the modulation.

Assume that the rotation number ω is strongly irrational, so that a corresponding invariant circle exists for $0<K\leq K(\omega)$. This is, for example, true if ω satisfies a Diophantine condition, i.e. there exist $\mu\geq 2$ and $\gamma>0$ such that for all $p,q\in\mathbb{Z}$, $q\neq 0$, $|\omega-p/q|\geq\gamma q^{-\mu}$. Then the map is analytically conjugated to the rotation

$$z_{n+1}=\exp(2\pi i\omega)z_n \qquad z_n=K\exp(2\pi i n\omega) \tag{2}$$

i.e., there exists a function $\Phi(z)$, holomorphic in a disk of radius $K(\omega)$ around $z=0$, such that $q_n=\Phi(z_n)$ and $p_n=\Phi(z_{n+1})-\Phi(z_n)$. The conjugation function $\Phi(z)$ can be expanded into a convergent power series $\Phi(z)=\sum_{n=1}^{\infty}\Phi_n z^n$ and must satisfy the functional equation

$$(\Phi(z)-1)(\delta^2\Phi)(z)=z \tag{3}$$

where $(\delta^2\Phi)(z)=\Phi(\exp(2\pi i\omega z)-2\Phi(z)+\Phi(\exp(-2\pi i\omega)z)$. The coefficients Φ_n are easily obtained by matching powers in (3): $\Phi_1=1/D_1$ and for all $n\geq 2$

$$\Phi_n=\frac{1}{D_n}\sum_{l=1}^{n-1}\Phi_l\Phi_{n-l}D_{n-l} \tag{4}$$

where $\{D_n\}_{n=1}^{\infty}$ is the divisors sequence

$$D_n=[2\sin(\pi n\omega)]^2. \tag{5}$$

The critical function $K(\omega)$ is obtained by means of Hadamard's formula:

$$K(\omega)^{-1}=\limsup_{n\to\infty}|\Phi_n|^{1/n}.$$

In figure 1 we have plotted the critical function $K(\omega)$, obtained by Hadamard's formula applied to Φ_{500}, at 5000 uniformly distributed random rotation numbers $\omega\in\,]0,\frac{1}{2}[$. The self-similarity of this fractal function, which vanishes at all rationals and for some Liouville numbers, is evident and it is a common feature both of area-preserving and complex analytic maps (Percival 1982, Marmi 1988a, 1989). Percival and Vivaldi (1988), by means of a subtle analysis of the influence of the (small) divisors D_m on the convergence radius $K(\omega)$, identified the coefficients Φ_m with sums over suitably defined planar trees. Thus they succeeded in writing the critical function as a solution of a transcendental equation and give a fairly good approximation of figure 1, when the points too close to the resonances occurring at rational ω are discarded.

We will now show how a very simple function, the Brjuno function, which only depends on the continued fraction expansion $[a_0,a_1,\ldots,a_n,\ldots]$ of ω, provides a good approximation of $K(\omega)$ with the right behaviour near resonances. The Brjuno function has also been used recently by Yoccoz (1988) in his study of the optimal number theoretical conditions for the Siegel centre theorem.

Let $\langle\ \rangle$ and $\|\ \|$ denote respectively the nearest integer and the distance from the nearest integer of a real number, and let $\omega\in\mathbb{R}\backslash\mathbb{Q}$. Consider two sequences $\{b_n\}_{n=0}^{\infty}\subseteq N$ and $\{\theta_n\}_{n=0}^{\infty}\subseteq\,]0,\frac{1}{2}[$ defined by $b_0=\langle\omega\rangle$, $\theta_0=\|\omega\|$ and for $n\geq 1$

$$b_n=\langle\theta_{n-1}^{-1}\rangle \qquad \theta_n=\|\theta_{n-1}^{-1}\|. \tag{6}$$

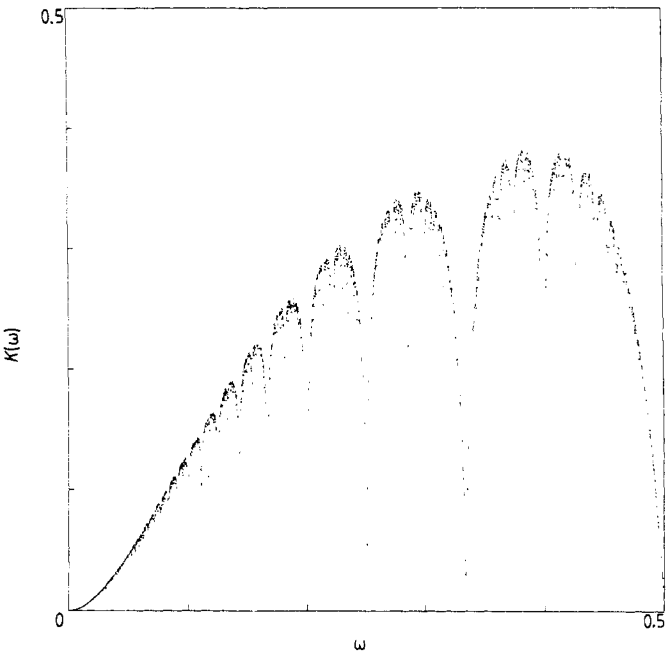

Figure 1. The critical function $K(\omega)$ for the MSM as a function of $\omega\in[0,\frac{1}{2}]$ computed by applying Hadamard's formula to Φ_{500}. The numerical error is approximately 10^{-3}.

Moreover, let $\beta_n=\Pi_{i=0}^{n}\theta_i$ if $n\geqslant 0$, $\beta_{-1}=1$. The values of β_j are related to the continued fraction approximations of ω since one can show that to each j corresponds $n=n(j)\geqslant j$ such that

$$\beta_j=(-1)^n(q_n\omega-p_n) \tag{7}$$

where p_n/q_n is a partial fraction of ω:

$$\frac{p_n}{q_n}=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_n}}}} \tag{8}$$

The Brjuno function $B(\omega)$ is defined by

$$B(\omega)=-\sum_{n=0}^{\infty}\beta_{n-1}\log\theta_n \tag{9}$$

and for all $\omega\in\mathbb{R}\backslash\mathbb{Q}\cap\,]0,\frac{1}{2}[$ satisfies

$$B(\omega)=-\log\omega+\omega B(\omega^{-1}). \tag{10}$$

In fact B converges for all Diophantine ω and for a class of Liouville numbers (those for which $\Sigma_{k=1}^{\infty}(\log q_{k+1}/q_k)<\infty$).

If $B(\omega)<\infty$, one can prove that the critical function $K(\omega)$ of the MSM (1) satisfies

$$\log K(\omega)\geqslant C'-4B(\omega) \tag{11}$$

where C' is some constant independent of ω. In particular, this shows that the convergence of the Brjuno function is a sufficient condition for the existence of an invariant circle of the MSM with rotation number ω.

The proof, based on Siegel's version of the majorant series method (Siegel 1942) and its improvement due to Brjuno (1971), is rather elementary but long, and we will only sketch it: we refer to Marmi (1989) for the details.

For all $n \geqslant 1$, let

$$\varepsilon_n := |\sin(\pi n \omega)|^2. \tag{12}$$

Following Siegel, we introduce the two sequences

$$\sigma_1 = 1 \qquad \sigma_n = \sum_{l=1}^{n-1} \sigma_l \sigma_{n-l} \qquad n \geqslant 2 \tag{13}$$

$$\delta_1 = \frac{1}{\varepsilon_1} \qquad \delta_n = \frac{1}{\varepsilon_n} \max_{1 \leqslant j \leqslant n-1} \delta_j \delta_{n-j} \qquad n \geqslant 2. \tag{14}$$

As one can immediately check by induction, for all $n \geqslant 1$, that

$$\Phi_n \leqslant \sigma_n \delta_n \tag{15}$$

thus one has

$$-\log K(\omega) \leqslant \sup_{n \geqslant 1} \frac{1}{n} \log \Phi_n \leqslant \sup_{n \geqslant 1} \frac{1}{n} (\log \sigma_n + \log \delta_n). \tag{16}$$

The contribution from σ_n is contained in the trivial constant term in (11), since this sequence keeps track of the (trivial) contribution to the growth rate of Φ_n coming from the recurrence (4) when the small divisors are disregarded, i.e. setting $D_n = 1$ for all n. The sequence δ_n extracts from the series the small divisors contribution. By means of an elementary but clever counting argument (lemma 9 in Brjuno 1971) one can estimate the growth rate by means of the Brjuno function: for all $n \geqslant 1$

$$\frac{1}{n} \log \delta_n \leqslant 4B(\omega) + C'' \tag{17}$$

where C'' is some constant independent both of ω and n. From this (11) immediately follows.

The Siegel theorem on the linearisation of complex analytic maps in a neighbourhood of an irrational indifferent fixed point leads to a recurrence for the power series coefficients of the linearisation function which is very similar to the recurrence (4) for the MSM conjugation function.

Yoccoz's study of this related problem shows that the estimates obtained by the method of Brjuno are not optimal with respect to the factor 4 in (11). In the Siegel problem the small divisors have the form

$$\hat{\varepsilon}_n = \sqrt{\varepsilon_n} = |\sin(\pi n \omega)|. \tag{18}$$

They contribute to the growth rate of the majorant series of the linearisation function through a sequence $\hat{\delta}_n$ which is defined as in (14), replacing ε_n with $\hat{\varepsilon}_n$. Therefore, by the same argument used above, estimate (17) holds true when δ_n is replaced with $\hat{\delta}_n$ and $4B(\omega)$ with $2B(\omega)$. All in all, one finds that

$$\log \hat{K}(\omega) \geqslant \hat{C} - 2B(\omega) \tag{19}$$

where $\hat{C}$ is a constant independent of ω and $\hat{K}(\omega)$ is the Siegel critical function (in this case the radius of convergence of the linearisation).

Yoccoz proves the existence of a positive constant $\hat{C}'$ such that for all ω for which $B(\omega)<\infty$, one has

$$|\log \hat{K}(\omega)+B(\omega)| \leq \hat{C}' \tag{20}$$

i.e. the factor 2 in the right-hand side of (19) is not optimal. This analysis suggests that the factor 4 on the right-hand side of (11) can be replaced by 2, so that one is naturally led to conjecture that

$$K(\omega)=\frac{\exp(-2B(\omega))}{C(\omega)} \tag{21}$$

where $C(\omega)$ is bounded away from 0 and ∞ uniformly in ω.

In figure 2 we have plotted $\exp(-2B(\omega))$, computed by the first 30 terms of the series (9), at the same random ω of figure 1. The close similarity with figure 1 is evident.

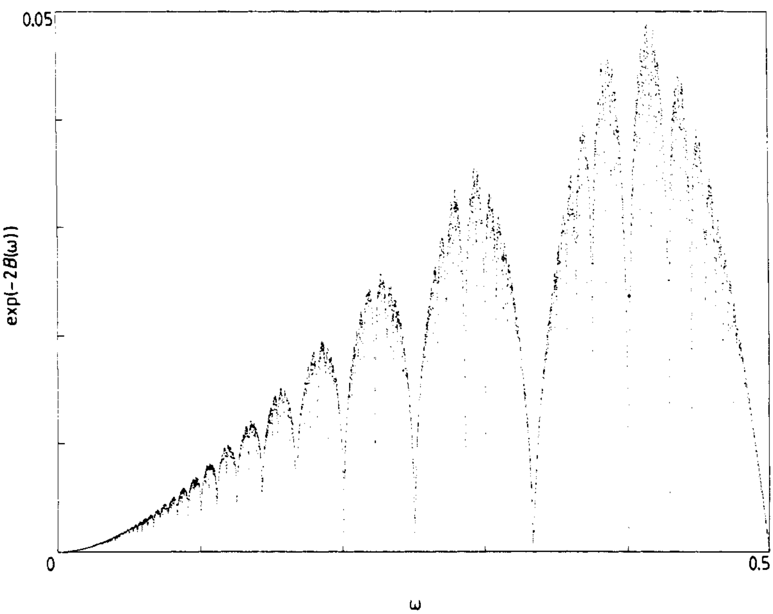

Figure 2. $\exp(-2B(\omega))$ at the same random ω of figure 1.

In figure 3 we exhibit the ratio $C(\omega)=\exp(-2B(\omega))/K(\omega)$: we needed to discard no points, since the Brjuno function has clearly correctly extracted the right vanishing rate of $K(\omega)$ at resonances. Note, for instance, that for $\omega \sim 1/n$, $n \geq 2$, one has corners, and that it seems to be possible that $C(\omega)$ can be extended to a continuous function on $]0, \frac{1}{2}[$.

As a further check we have considered a special class of rotation numbers, namely those quadratic irrationals whose continued fraction is constant, $\omega_p=[p, p, \ldots]$, $p \geq 1$. For these irrationals one can easily compute exactly the Brjuno function directly from (10). In fact $B(\omega)$ is invariant under the action of integer translations $\omega \mapsto \omega+m$, $m \in \mathbf{Z}$,

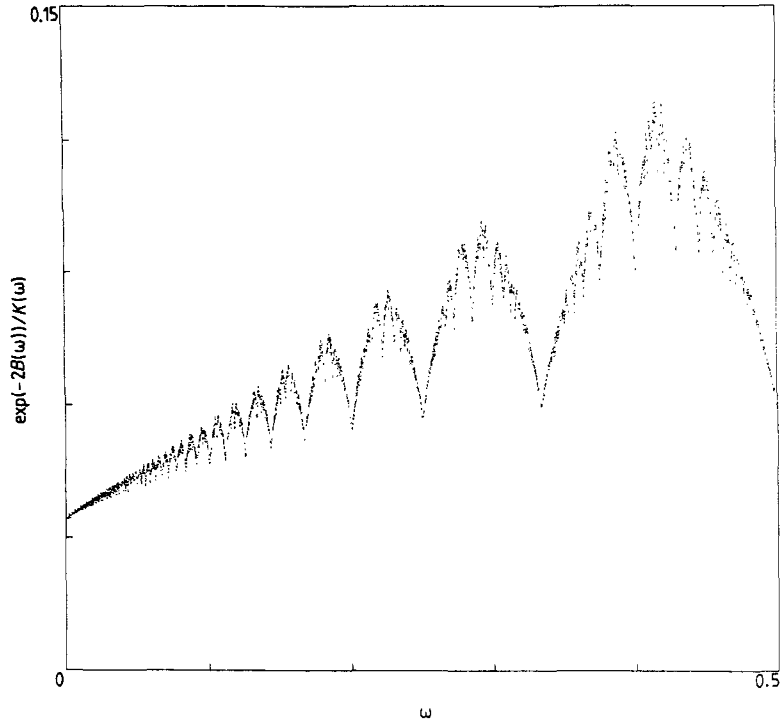

Figure 3. The ratio $C(\omega)=\exp(-2B(\omega))/K(\omega)$ at the same random ω of figure 1 and figure 2.

and the quadratic irrationals

$$\omega_p=\frac{\sqrt{p^2+4}-p}{2}=[p,p,\ldots] \tag{22}$$

are solutions of the equation

$$\omega_p+p=1/\omega_p. \tag{23}$$

Thus by (10) one has immediately

$$B(\omega_p)=\frac{-\log\omega_p}{1-\omega_p}. \tag{24}$$

In figure 4 we have plotted the critical function $K(\omega_p)$ as a function of $1/p^2$ for $20\leqslant p\leqslant 500$. Clearly in the limit $p\to\infty$, $K(\omega_p)\simeq 1/p^2$, in agreement with (12) since $\omega_p\simeq 1/p$ and $\exp(-2B(\omega_p))=\omega_p^{2/(1-\omega_p)}\simeq 1/p^2$.

To conclude, we want to stress that from our analysis it is evident that the Brjuno function also provides a simple and quite accurate tool for obtaining *a priori* estimates of breakdown thresholds, once one has fixed a normalisation by means of a single computation of $K(\omega)$ from the perturbative series, so as to fix the value of $C(\omega)$ at some $\omega=\omega^*$. Indeed, from figure 3 one sees that $C(\omega)$ is uniformly bounded away from 0 and ∞ and only undergoes a variation through a factor of about three on the internal $]0,\frac{1}{2}[$. A more careful inspection shows that the maximum is reached at $\omega=\omega_2=\sqrt{2}-1$, where $C(\omega_2)\simeq 0.130\,29$, thus $K(\omega)\geqslant C(\omega_2)^{-1}\exp(-2B(\omega))\simeq 7.6752\exp(-2B(\omega))$ provides a lower bound, especially sharp for ω close to ω_2.

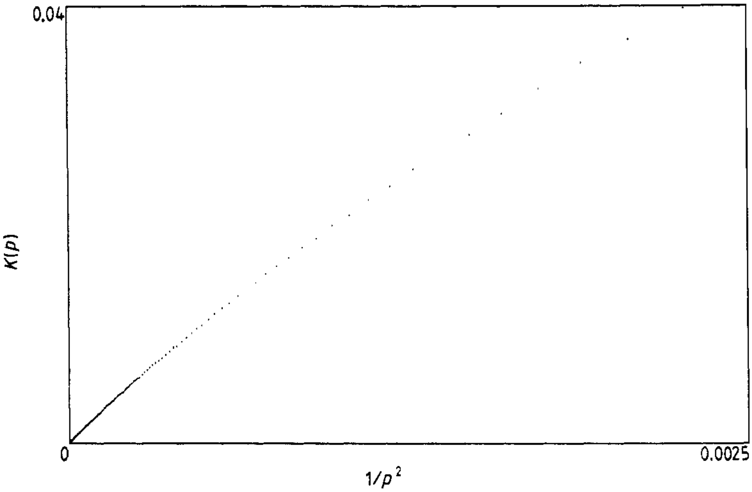

Figure 4. The critical function $K(\omega_p)$ as a function of $1/p^2$; $\omega_p = [p, p, \ldots] \frac{1}{2}(\sqrt{p^2+4} - p)$, and $20 \leq p \leq 500$.

Similar results have been obtained for other complex area-preserving maps, like the semistandard map (Percival 1982, Buric *et al* 1989, Marmi 1989), and we expect them to be possibly valid for more general classes of maps.

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